Supplemental Material

Paper ID: 518

A Appendix

A.1 Convex Relaxation and Dual of Problem (11)

Since problem (11) is a mixed integer problem regarding $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$, it is hard to directly optimize. Motivated by (Tan *et al.* 2010), we apply convex relaxation and Lagrange dual to make some transformations.

Firstly, we introduce dual variables $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{N_h}$ for the hingle loss constraint:

$$\ell(\boldsymbol{\mu}, \boldsymbol{\eta}; (\mathbf{x}_i, y_i)) = \max(0, 1 - y_i \boldsymbol{\mu}^T (\boldsymbol{\eta} \odot \mathbf{x}_i)). \quad (21)$$

As to problem (11), we can get the Lagrangian function of the inner problem w.r.t $\boldsymbol{\mu}$:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\mu}, \ell, \boldsymbol{\alpha}, \boldsymbol{\beta}) &= \frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu_{h-1}})^T \boldsymbol{\Sigma}_h(\boldsymbol{\eta})^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu_{h-1}}) \\ &+ \frac{C}{q}\sum_{i=1}^{N_h} D_i \ell(\boldsymbol{\mu}, \boldsymbol{\eta}; (\mathbf{x}_i, y_i))^q + \sum_{i=1}^{N_h}(-\boldsymbol{\beta}_i \ell(\boldsymbol{\mu}, \boldsymbol{\eta}; (\mathbf{x}_i, y_i))) \\ &+ \sum_{i=1}^{N_h} \alpha_i (1 - y_i(\boldsymbol{\mu} \odot \boldsymbol{\eta})^T \mathbf{x}_i - \ell(\boldsymbol{\mu}, \boldsymbol{\eta}; (\mathbf{x}_i, y_i))) \end{aligned} \quad (22)$$

By taking derivative over variables $\boldsymbol{\mu}$ and ℓ, we get:

$$\begin{aligned} &\nabla_{\boldsymbol{\mu}} \mathcal{L} = \boldsymbol{\Sigma}(\boldsymbol{\eta})^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_{h-1}) - \sum_{i=1}^{N_h} \alpha_i y_i (\mathbf{x}_\mathbf{i} \odot \boldsymbol{\eta}) = 0, \\ &\nabla_{\ell_i} \mathcal{L} = CD_i - \alpha_i - \beta_i = 0, \alpha_i, \beta_i \geq 0, \text{ for q=1 }, \\ &\nabla_{\ell_i} \mathcal{L} = CD_i \ell_i - \alpha_i - \beta_i = 0, \beta_i = 0, \text{ for q=2 }. \end{aligned}$$

With some transformations, then:

$$\begin{aligned} &\boldsymbol{\mu} - \boldsymbol{\mu}_{h-1} = \boldsymbol{\Sigma}(\boldsymbol{\eta}) \sum_{i=1}^{N_h} \alpha_i y_i (\mathbf{x}_i \odot \boldsymbol{\eta}), \\ &0 \leq \alpha_i \leq CD_i, \text{ for q=1 }, \\ &\ell_i = \alpha_i/(CD_i), \text{ for q=2 }. \end{aligned}$$

Let $\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}) := \sum_{i=1}^{N_h} \alpha_i y_i (\mathbf{x}_i \odot \boldsymbol{\eta})$, $\mathcal{A} := \{\boldsymbol{\alpha} \in \mathbb{R}^{N_h} | 0 \leq \alpha_i \leq U, \forall i \in [N_h]\}$ is the domain of $\boldsymbol{\alpha}$ (here, $U = CD_i$ for $q = 1$ and $U = \infty$ for $q = 2$), then we can get the dual of inner problem (11) as:

$$\begin{aligned} \max_{\boldsymbol{\alpha} \in \boldsymbol{\Lambda}} &- \frac{1}{2}\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta})^T \boldsymbol{\Sigma}(\boldsymbol{\eta}) \mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}) - \frac{q-1}{2C}\sum_{i=1}^{N_h} \frac{\alpha_i^2}{D_i} \\ &+ \sum_{i=1}^{N_h} \alpha_i - \boldsymbol{\mu}_{h-1}^T \mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}), \end{aligned} \quad (23)$$

We define objective of (23) as $f(\boldsymbol{\alpha}, \boldsymbol{\eta})$ for convenience. Problem (11) can be reformulated as a minmax problem:

$$\min_{\boldsymbol{\eta} \in \boldsymbol{\Lambda}} \max_{\boldsymbol{\alpha} \in \mathcal{A}} f(\boldsymbol{\alpha}, \boldsymbol{\eta}), \quad (24)$$

This problem is also a mixed integer problem, but we have the following property according to minmax inequality (Sion 1958):

$$\min_{\boldsymbol{\eta} \in \boldsymbol{\Lambda}} \max_{\boldsymbol{\alpha} \in \mathcal{A}} f(\boldsymbol{\alpha}, \boldsymbol{\eta}) \geq \max_{\boldsymbol{\alpha} \in \mathcal{A}} \min_{\boldsymbol{\eta} \in \boldsymbol{\Lambda}} f(\boldsymbol{\alpha}, \boldsymbol{\eta}), \quad (25)$$

The latter problem of (25) provides a lower bound to problem (24) and it is also a convex problem. By introducing a variable θ, we can transform the problem into :

$$\max_{\theta \in \mathbb{R}, \boldsymbol{\alpha} \in \mathcal{A}} \theta, \text{ s.t. } \theta \leq f(\boldsymbol{\alpha}, \boldsymbol{\eta}), \forall \eta \in \boldsymbol{\Lambda}. \quad (26)$$

A.2 Solving for the Primal of Problem (15)

We prove that problem (16) is the primal of problem (15).

Let $\Omega(\mathbf{w}) = \frac{1}{2}(\sum_{k=1}^{K} \|\mathbf{w}_k - \mathbf{w}_k^{h-1}\|)^2$. Define second order crone $\mathcal{Q}_r = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{r+1}, \|\mathbf{u}\|_2 \leq v\}$. Let $\mathbf{z}_k = \|\mathbf{w}_k - \mathbf{w}_k^{h-1}\|$, then $\Omega(\mathbf{w}) = \frac{1}{2}\mathbf{z}^2$, where $\mathbf{z} = \sum_{k=1}^{K} \mathbf{z}_k, \mathbf{z}_k \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$. Then problem (16) can be reformulated as:

$$\begin{aligned} \min_{\mathbf{z}, \mathbf{w}_k, \ell} &\frac{1}{2}\mathbf{z}^2 + \frac{C}{q}\sum_{i=1}^{N_h} D_i \ell(\mathbf{w}_k, \boldsymbol{\eta})^q, \\ \text{s.t. } &\sum_{k=1}^{K} \mathbf{z}_k \leq \mathbf{z}, (\mathbf{w}_k - \mathbf{w}_k^{h-1}, \mathbf{z}_k) \in \mathcal{Q}_r, \end{aligned} \quad (27)$$

We introduce $\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \boldsymbol{\delta}, \boldsymbol{\epsilon}$ here. With $\boldsymbol{\delta}, \boldsymbol{\epsilon}$ and the constraints on second order crone $\mathcal{Q}_r$, we point out that $\boldsymbol{\delta}_k^T(\mathbf{w}_k -$

$\mathbf{w}_k^{h-1}) + \epsilon_k \mathbf{z}_k$ along with $\|\boldsymbol{\delta}_k\| \leq \epsilon_k$ equals original constraints on $\mathcal{Q}_r$ with Lagrangian multiplier. Now Lagrangian function can be written as:

$$\mathcal{L}(\mathbf{z}, \mathbf{w}_k, \ell, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \boldsymbol{\delta}, \boldsymbol{\epsilon}) = \frac{1}{2}\mathbf{z}^2 + \frac{C}{q}\sum_{i=1}^{N_h} D_i \ell_i^q + \sum_{i=1}^{N_h}(-\beta_i \ell_i)$$
$$- \sum_{k=1}^{K}(\boldsymbol{\delta}_k^T(\mathbf{w}_k - \mathbf{w}_k^{h-1}) + \epsilon_k \mathbf{z}_k) + \gamma(\sum_{k=1}^{K}\mathbf{z}_k - \mathbf{z})$$
$$+ \sum_{i=1}^{N_h}\alpha_i(1 - y_i\sum_{k=1}^{K}\mathbf{w}_k^T\widehat{\mathbf{x}}_i^k - \ell_i). \quad (28)$$

Taking derivatives w.r.t $\mathbf{z}, \mathbf{w}_k, \ell_i$, the KKT condition is as follows:

$$\nabla_{\mathbf{z}}\mathcal{L} = \mathbf{z} - \gamma = 0,$$
$$\nabla_{\mathbf{z}_k}\mathcal{L} = \gamma - \epsilon_k = 0,$$
$$\nabla_{\mathbf{w}_k}\mathcal{L} = -\sum_{i=1}^{N_h}\alpha_i y_i \widehat{\mathbf{x}}_i^k - \boldsymbol{\delta}_k = 0,$$
$$\nabla_{\ell_i}\mathcal{L} = CD_i - \alpha_i - \beta_i = 0, \alpha_i, \beta_i \geq 0, \text{ for q=1 },$$
$$\nabla_{\ell_i}\mathcal{L} = CD_i\ell_i - \alpha_i - \beta_i = 0, \beta_i = 0, \text{ for q=2 },$$
$$\|\boldsymbol{\delta}_k\| \leq \epsilon_k.$$

Substituting all the equations back into Lagrangian function, we have

$$\mathcal{L}(\mathbf{z}, \mathbf{w}_k, \ell, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \boldsymbol{\delta}, \boldsymbol{\epsilon}) = -\frac{1}{2}\gamma^2 - \frac{q-1}{2C}\sum_{i=1}^{N_h}\frac{\alpha_i^2}{D_i} + \sum_{i=1}^{N_h}\alpha_i$$
$$+ \sum_{k=1}^{K}\boldsymbol{\delta}^k \mathbf{w}_k^{h-1}. \quad (29)$$

Let $\mathcal{A} := \{\boldsymbol{\alpha} \in \mathbb{R}^{N_h} | 0 \leq \alpha_i \leq U\}$ be the domain of $\boldsymbol{\alpha}$ where $U = CD_i$ for $q = 1$ and $U = \infty$ for $q = 2$. We then rewrite dual problem of Lagrangian:

$$\max_{\gamma\in\mathbb{R}, \boldsymbol{\alpha}\in\mathcal{A}} \mathcal{L}(\mathbf{z}, \mathbf{w}_k, \ell, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \boldsymbol{\delta}, \boldsymbol{\epsilon})$$
$$\text{s.t.} \|\sum_{k=1}^{K}\alpha_i y_i \widehat{\mathbf{x}}_i^k\| \leq \gamma, k = 1, \ldots, K \quad (30)$$

Let $\theta := \mathcal{L}(\mathbf{z}, \mathbf{w}_k, \ell, \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma, \boldsymbol{\delta}, \boldsymbol{\epsilon})$ and $\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}_k) = \sum_{k=1}^{K}\alpha_i y_i(\widehat{\mathbf{x}}_i \odot \boldsymbol{\eta}_k)$. We further define $f(\boldsymbol{\alpha}, \boldsymbol{\eta}_k) = -\frac{1}{2}\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}_k)^T\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}_k) - \frac{q-1}{2C}\sum_{i=1}^{N_h}\frac{\alpha_i^2}{D_i} + \sum_{i=1}^{N_h}\alpha_i - (\mathbf{w}_k^{h-1})^T\mathbf{g}(\boldsymbol{\alpha}, \boldsymbol{\eta}_k)$. Then,

$$\max_{\theta, \boldsymbol{\alpha}\in\mathcal{A}} \theta, \text{ s.t. } \theta \leq f(\boldsymbol{\alpha}, \boldsymbol{\eta_k}), k = 1, \ldots, K. \quad (31)$$

Since $\widehat{\mathbf{x}}_i = \boldsymbol{\Sigma}_k^{\frac{1}{2}}\mathbf{x}_i$, with some transformation, we can get that (31) is equivalent to (15).

A.3 Conjugate Dual of Problem (17)

Problem (17) can be written as:

$$\min_{\mathbf{w}} \Omega(\mathbf{w}) + C\sum_{i=1}^{N_h} L_i(\mathbf{w}^T\widehat{\mathbf{x}}_i). \quad (32)$$

Let $p_i := \mathbf{w}^T\widehat{\mathbf{x}}_i$, (32) can be reformulated as:

$$\min_{\mathbf{w}} \Omega(\mathbf{w}) + C\sum_{i=1}^{N_h} L_i(p_i), \text{ s.t. } p_i = \mathbf{w}^T\widehat{\mathbf{x}}_i, i = 1, \ldots, N_h \quad (33)$$

Now the Lagrangian function:

$$\mathcal{L}(\mathbf{w}, \mathbf{p}, \boldsymbol{\alpha}) = \Omega(\mathbf{w}) + C\sum_{i=1}^{N_h} L_i(\mathbf{p}_i) + C\sum_{i=1}^{N_h}\alpha_i(p_i - \mathbf{w}^T\widehat{\mathbf{x}}_i)$$
$$= \Omega(\mathbf{w}) - C\sum_{i=1}^{N_h}\alpha_i\mathbf{w}^T\widehat{\mathbf{x}}_i + C\sum_{i=1}^{N_h}(L_i(p_i) + \alpha_i p_i). \quad (34)$$

Let $\mathbf{z}(\boldsymbol{\alpha}) = C\sum_{i=1}^{N_h}\alpha_i\widehat{\mathbf{x}}_i$. If we decouple $\mathbf{w}$ and $\mathbf{p}$, and minimize Lagrangian function w.r.t $\mathbf{w}$ and $\mathbf{p}$, then:

$$\min_{\mathbf{w},\mathbf{p}} \mathcal{L}(\mathbf{w}, \mathbf{p}, \boldsymbol{\alpha})$$
$$= \min_{\mathbf{w}}(\Omega(\mathbf{w}) - \mathbf{w}^T\mathbf{z}(\boldsymbol{\alpha})) + \min_{\mathbf{p}} C\sum_{i=1}^{N_h}(L_i(p_i) + \alpha_i p_i)$$
$$= -\max_{\mathbf{w}}(\mathbf{w}^T\mathbf{z}(\boldsymbol{\alpha}) - \Omega(\mathbf{w})) - \max_{\mathbf{p}} C\sum_{i=1}^{N_h}(-\alpha_i p_i - L_i(p_i))$$
$$= -\Omega^*(\mathbf{z}(\boldsymbol{\alpha})) - C\sum_{i=1}^{N_h} L_i^*(-\alpha_i). \quad (35)$$

Thus the dual problem is:

$$\max_{\boldsymbol{\alpha}\geq\mathbf{0}} -\Omega^*(\mathbf{z}(\boldsymbol{\alpha})) - C\sum_{i=1}^{N_h} L_i^*(-\alpha_i). \quad (36)$$

A.4 Computation of $\nabla^*\Omega(\mathbf{z}(\boldsymbol{\alpha}))$

In order to solve $\mathbf{w}$, we need to compute $\mathbf{w} = \nabla^*\Omega(\mathbf{z})$ given $\mathbf{z}$ and $\mathbf{w}_{h-1}$. Based on the conjugate dual property, we have the following problem:

$$\mathbf{w} = \arg\max_{\mathbf{w}} \mathbf{w}^T\mathbf{z} - \Omega(\mathbf{w})$$
$$= \arg\max_{\mathbf{w}} \mathbf{w}^T\mathbf{z} - \frac{\sigma}{2}\|\mathbf{w} - \mathbf{w}_{h-1}\|^2 - \frac{1}{2}(\sum_{k=1}^{K}\|\mathbf{w}_k - \mathbf{w}_k^{h-1}\|^2)$$
$$= \arg\max_{\mathbf{w}} -\frac{\sigma}{2}\|\mathbf{w} - \mathbf{w}_{h-1} - \frac{\mathbf{z}}{\sigma}\|^2 - \frac{1}{2}(\sum_{k=1}^{K}\|\mathbf{w}_k - \mathbf{w}_k^{h-1}\|^2)$$
$$= \arg\min_{\mathbf{w}} \frac{\sigma}{2}\|\mathbf{w} - \mathbf{w}_{h-1} - \frac{\mathbf{z}}{\sigma}\|^2 + \frac{1}{2}(\sum_{k=1}^{K}\|\mathbf{w}_k - \mathbf{w}_k^{h-1}\|^2) \quad (37)$$

Problem (37) is strictly convex problem, thus a unique minimizer exits, and can be computed in close-form. According to (Martins *et al.* 2011), we give the detailed solution as shown in Algorithm 3.

Algorithm 3 Computation of $\mathbf{w} = \nabla^*\Omega(\mathbf{z})$

Require: $\mathbf{z}, \mathbf{w}_{h-1}$, parameter $\frac{1}{\sigma}$.
Initialize $\boldsymbol{\omega} = \frac{z}{\boldsymbol{\sigma}}$.
Compute $\widehat{o}_k = \|\boldsymbol{\omega}_k\|$ where $\boldsymbol{\omega}_k$ is associated with $\mathbf{w}_k$ for $k = 1, \ldots, K$
Sort $\widehat{\mathbf{o}}$ to obtain $\bar{\mathbf{o}}$ such that $\bar{o}_1 \geq \cdots \geq \bar{o}_K$.
Find $\rho = \max\{k|\bar{o}_k - \frac{s}{1+ks}\sum_{i=1}^{k}\bar{o}_i > 0, k = 1, \ldots, K\}$.
Compute a threshold value $\zeta = \frac{s}{1+\rho s}\sum_{i=1}^{\rho}\bar{o}_i$.
Calculate $\mathbf{o}$, where $o_k = \begin{cases}\widehat{o}_k - \zeta, & \text{if } \widehat{o}_k > \zeta,\\ 0, & \text{Otherwise.}\end{cases}$
Calculate $\widehat{\boldsymbol{\omega}}_k = \begin{cases}\frac{o_k}{\|\boldsymbol{\omega}_k\|}\boldsymbol{\omega}_k, & \text{if } o_k > 0,\\ 0, & \text{Otherwise.}\end{cases}$
Let $\mathbf{w} = [\widehat{\boldsymbol{\omega}}_k]_{k=1}^{K}$ and return $\mathbf{w}$.

A.5 Online Update of Imbalance Measures

In this paper, we focus on three performance measures: F-measure, AUROC and AUPRC instead of mistake number or classification loss used in traditional methods. According to Algorithm 2, we need to maintain and update the performance measures M_{h+1}^j at each iteration h. However, if we directly compute M_{h+1}^j, it is computation expensive and requires to store all historical predictions and labels, which is really inefficient. Instead, we present how to update M_{h+1}^j by only using M_h and current $\mathbf{f_h}, \mathbf{y}_h$.

For F-measure, let $\bar{\mathbf{y}}_h = (\mathbf{y}_h + 1)/2$ and $\widehat{\mathbf{y}}_h = \text{sign}(\mathbf{f}_h > 0)$, $a_h = \sum_{\tau=1}^{h}\bar{\mathbf{y}}_\tau \cdot \widehat{\mathbf{y}}_\tau$, $c_h = \sum_{\tau=1}^{h}\sum\bar{\mathbf{y}}_\tau + \sum_{\tau=1}^{h}\sum\widehat{\mathbf{y}}_\tau$. We can calculate F-measure as: $F_{h+1} = \frac{2a_h}{c_h}$. In order to compute F-measure incrementally, we only need to update a_h and c_h as:

$$a_{h+1} = a_h + \bar{\mathbf{y}}_{h+1} \cdot \widehat{\mathbf{y}}_{h+1},$$
$$c_{h+1} = c_h + \sum\bar{\mathbf{y}}_{h+1} + \sum\hat{\mathbf{y}}_{h+1}.$$

AUROC and AUPRC are different from F-measure in that they need to compute the area value under various thresholds. We introduce two auxiliary hash table L_+^h and L_-^h of size m that partition $(0,1)$ into m uniform ranges. For $i \in \{1, \ldots, m\}$, $L_+^h[i]$ stores the number of positive examples before (including) h-th iteration with predictions f such that $\sigma(f) \in [(i-1)/m, i/m)$. Similarly, L_-^h stores negative examples with $\sigma(f) \in [(i-1)/m, i/m)$. σ is the sigmoid function that normalize f to $(0,1)$. Let N_h^+ and N_h^- denote the number of positive and negative examples respectively. We then compute the True Positive Rate (TPR) and False Positive Rate (FPR) as: $\text{TPR}(i) = \sum_{j=i+1}^{m} L_+^h[j]/N_h^+$ and $\text{FPR}(i) = \sum_{j=i+1}^{m} L_-^h[j]/N_h^-$. Thus,

$$\text{AUROC} = \frac{1}{2}\sum_{i=0}^{m-1}[\text{FPR}(i{+}1){-}\text{FPR}(i)][\text{TPR}(i){+}\text{TPR}(i{+}1)].$$

Similarly, Precision (P) and Recall (R) are computed as: $\text{P}(i) = \sum_{j=i+1}^{m} L_+^h[j]/\sum_{j=i+1}^{m}(L_h^+[j] + L_h^-[j])$ and $\text{R}(i) = \text{TPR}(i)$. Similarly,

$$\text{AUPRC} = \frac{1}{2}\sum_{i=0}^{m-1}[\text{R}(i) - \text{R}(i+1)][\text{P}(i) + \text{P}(i+1)].$$

In order to compute AUROC and AUPRC incrementally, we only need to maintain and update L_+^h and L_-^h.

A.6 Proof of Proposition 1

We mainly consider F1-score, whose computation is

$$F(h) = \frac{2(P_1 - \texttt{fn})}{2P_1 - \texttt{fn} + \texttt{fp}},$$

where h can be any hypothesis (classifiers, models, etc.), fn and fp denote the false negative probability and false positive probability, respectively.

Following (Parambath *et al.* 2014), we define the following notations for binary classification:

$$\boldsymbol{a}(\theta) = [1 - \frac{\theta}{2}, \frac{\theta}{2}] \in \mathbb{R}^2,$$
P_1 : the marginal probability of the positive instances,
$\boldsymbol{e} = [\texttt{fn}, \texttt{fp}] \in \mathbb{R}^2$: error profile,
$\boldsymbol{E}(h) = [\texttt{fn}, \texttt{fp}] \in \mathbb{R}^2$: error profile of h,
$$\boldsymbol{e}(\theta) \in \arg\min_{\boldsymbol{e}' \in \mathcal{E}}\langle \boldsymbol{a}(\theta), \boldsymbol{e}'\rangle.$$

Lemma 1. (Proposition 4 in (Parambath *et al.* 2014)) Let $F^* = \max_{\boldsymbol{e}' \in \mathcal{E}(\mathcal{H})} F(\boldsymbol{e}')$. We have:

$$\boldsymbol{e} \in \arg\min_{\boldsymbol{e}' \in \mathcal{E}(\mathcal{H})}\langle \boldsymbol{a}(F^*), \boldsymbol{e}'\rangle \Leftrightarrow F(\boldsymbol{e}) = F^*.$$

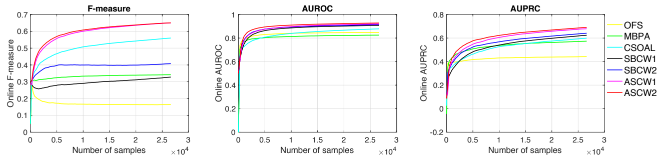
Lemma 2. (In the proof of Proposition 6 in (Parambath *et al.* 2014)) $F(\boldsymbol{e}) = t \Leftrightarrow \langle a(t), \boldsymbol{e}\rangle = \min_{\boldsymbol{e}' \in \mathcal{E}(\mathcal{H})}\langle \boldsymbol{a}(t), \boldsymbol{e}'\rangle = \frac{2P_1(t-1)}{2}$.

Here we re-present Proposition 1 as follows:

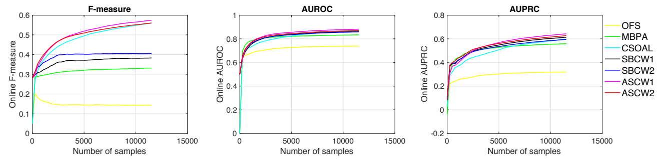
Proposition 1. Given the evenly distributed values $\theta_1, ..., \theta_K$ and the cost vector $\boldsymbol{a}(\theta) = [1 - \frac{\theta}{2}, \frac{\theta}{2}]$, let $\Delta = \frac{\theta_j - \theta_{j+1}}{2} = \frac{1}{2K}$. Denote $F^* = \max_{\boldsymbol{e}} F(\boldsymbol{e})$ the maximum F-measure and $F(\boldsymbol{\mu})$ a function of $\boldsymbol{\mu}$ that computes the F-measure achieved by $\boldsymbol{\mu}$. Assume that $\{\boldsymbol{\mu}_h^1, ..., \boldsymbol{\mu}_h^K\}$ minimizes the cost-sensitive loss to a certain degree and $\boldsymbol{E}(\boldsymbol{\mu}) = [\texttt{fn}, \texttt{fp}]$, i.e., false negative probability and false positive probability. Then the F-measure achieved by Algorithm 2 has the following lower bound as long as h increases:

$$\max_{j=1,\ldots,K} F(\boldsymbol{\mu}_h^j) \geq F^* - \Delta - \frac{\epsilon_0}{P_1},$$

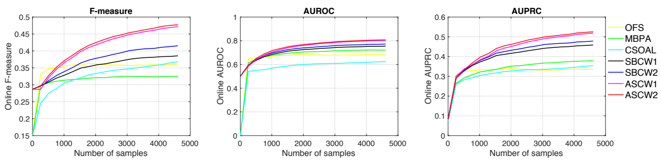
where $k = \arg\max_{j=1,\ldots,K} F(\boldsymbol{\mu}_h^j)$ and $\langle \boldsymbol{a}(\theta_j), \boldsymbol{E}(\boldsymbol{\mu}_h^k)\rangle \leq \min_{\boldsymbol{\mu}}\langle \boldsymbol{a}(\theta_j), \boldsymbol{E}(\boldsymbol{\mu})\rangle + \epsilon_0$.



(a) real-sim with ratio 1:5

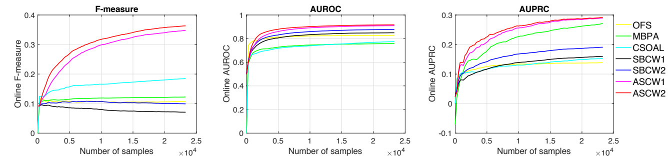


(b) rcv1 with ratio 1:5

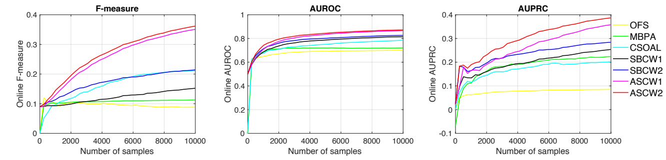


(c) news20 with ratio 1:5

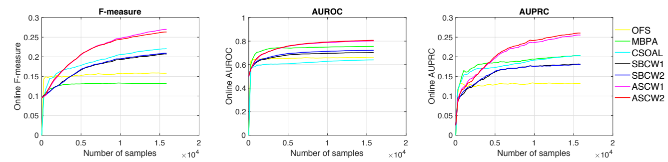
Figure S1: Online performance for ratio 1:5



(a) real-sim with ratio 1:20



(b) rcv1 with ratio 1:20



(c) news20 with ratio 1:19

Figure S2: Online performance for ration 1:20 (1:19)

Proof.

$$
\begin{aligned}
& F^* - F(\boldsymbol{\mu}_h^j) \\
=& F^* - \frac{2(P_1 - \boldsymbol{E}(\boldsymbol{\mu}_h^j))}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
=& \frac{2P_1(F^* - 1) + \langle \boldsymbol{a}(\theta^*), \boldsymbol{E}(\boldsymbol{\mu}_h^j) \rangle}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
=& \frac{\langle \boldsymbol{a}(\theta^*) - \boldsymbol{a}(\theta_j), \boldsymbol{E}(\boldsymbol{\mu}_h^j) \rangle + \langle \boldsymbol{a}(\theta_j), \boldsymbol{E}(\boldsymbol{\mu}_h^j) + 2P_1(\theta^* - 1) \rangle}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
=& \frac{(\theta^* - \theta_j) + \langle \boldsymbol{a}(\theta_j), \boldsymbol{E}(\boldsymbol{\mu}_h^j) \rangle + 2P_1(\theta^* - 1)}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
\leq& \frac{(\theta^* - \theta_j) + \langle \boldsymbol{a}(\theta_j), e(\theta_j) \rangle + \epsilon_0 + 2P_1(\theta^* - 1)}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
=& \theta^* - \theta_j + \frac{\epsilon_0}{2P_1 - \boldsymbol{E}_1(\boldsymbol{\mu}_h^j) + \boldsymbol{E}_2(\boldsymbol{\mu}_h^j)} \\
\leq& \Delta + \frac{\epsilon_0}{P_1}. \qquad (1)
\end{aligned}
$$

□

A.7 Additional Results

In Figure 1, we only show the online performance with respect to the ratio 1:10. Here we show other imbalance ratios in Figure S1 and Figure S2.

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